Department of Mathematical and Computational Sciences

Analysis I

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(Dedications)

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1 The Construction of The Real Numbers

1.1 The Incompleteness of The Rationals

$\mathbb{Q}$ is not complete ..., there are holes. Here's an example.

1.2 Dedekind Cuts and Addition

1.3 Multiplication of Dedekind Cuts

Throughout, let α, β, γ denote three real numbers.

Proposition 1.3.1: Neutrality of 0

$\alpha \cdot \mathbf{0} = \mathbf{0}$.

Proof. Suppose first that $\alpha \geq \mathbf{0}$ and write $\alpha = A|A'$. By definition, the lower set of the product $\alpha \cdot \mathbf{0}$ is $\{xy : x \in A \cap \mathbb{Q}_{\geq 0}, y \in \mathbb{Q}_{<0} \cap \mathbb{Q}_{\geq 0}\} \cup \mathbb{Q}_{<0}$. As $\mathbb{Q}_{<0} \cap \mathbb{Q}_{\geq 0} = \varnothing$, it follows that the lower set of $\alpha \cdot \mathbf{0}$ is $\mathbb{Q}_{<0}$, and hence $\alpha \cdot \mathbf{0} = \mathbf{0}$. If $\alpha < \mathbf{0}$, then $-\alpha > \mathbf{0}$ and so by definition

$$\alpha \cdot \mathbf{0} = -((-\alpha) \cdot \mathbf{0}) = -\mathbf{0}$$

By the uniqueness of additive inverses and additive identity property of addition, it follows that $-\mathbf{0} = \mathbf{0}$ and hence $\alpha \cdot \mathbf{0} = \mathbf{0}$. □

Proposition 1.3.2: Closure

$\alpha \cdot \beta$ is a real number

Proof. It suffices to prove this in the case that $\alpha, \beta > \mathbf{0}$. Note that by definition $-1 \in \alpha \cdot \beta$, and so $\alpha \cdot \beta \neq \varnothing$. Moreover as $\alpha, \beta \neq \mathbb{Q}$, we can find positive rational numbers $x \in \mathbb{Q} \setminus \alpha, y \in \mathbb{Q} \setminus \beta$. Thus $x > a$ for all $a \in \alpha$ and $y > b$ for all $b \in \beta$, so $xy > ab$ and hence $xy \notin \alpha \cdot \beta$, thus $\alpha \cdot \beta \neq \mathbb{Q}$. Now let $z \in \alpha \cdot \beta, q \in \mathbb{Q}$ and suppose $q < z$. If $q \leq 0$, then $q \in \alpha \cdot \beta$ immediately. If $q > 0$, then $0 < q < z$ and so there is $a \in \alpha \cap \mathbb{Q}_{>0}, b \in \beta \cap \mathbb{Q}_{>0}$ such that $z = ab$. Note that as $0 < q < z$, we have $0 < \frac{q}{z} < 1$, and so $0 < a\left(\frac{q}{z}\right) < a$, namely $a\left(\frac{q}{z}\right) \in \alpha$. Finally, we have

$$q = \frac{qz}{z} = \underbrace{\left[a\left(\frac{q}{z}\right)\right]}_{\in\alpha}\underbrace{b}_{\in\beta}$$

and so $q \in \alpha \cdot \beta$. Lastly, we show $\alpha \cdot \beta$ has no maximum. Let $z \in \alpha \cdot \beta$. If $z \leq 0$, as $\alpha, \beta > \mathbf{0}$, there is $a \in \alpha \cap \mathbb{Q}_{>0}, b \in \beta \cap \mathbb{Q}_{>0}$, and so $z \leq 0 < ab$, where $ab \in \alpha \cdot \beta$. If $z > 0$, by definition there is $a \in \alpha \cap \mathbb{Q}_{>0}$ and $b \in \beta \cap \mathbb{Q}_{>0}$ such that $z = ab$. Moreover as neither α nor β achieve their maximums, we can find $\widehat{a} \in \alpha, \widehat{b} \in \beta$ such that $\widehat{a} > a, \widehat{b} > b$. Thus, $z = ab < \widehat{a}\widehat{b}$ where $\widehat{a}\widehat{b} \in \alpha \cdot \beta$, and so $\alpha \cdot \beta$ is a real number. □

Proposition 1.3.3: Commutativity

$\alpha \cdot \beta = \beta \cdot \alpha$.

Proof. □

Lemma 1.3.4

$-(-\alpha) = \alpha$ and $(-\alpha) \cdot \beta = -(\alpha \cdot \beta)$.

Proof. Exercise. □

Proposition 1.3.5: Associativity

$\alpha \cdot (\beta \cdot \gamma) = (\alpha \cdot \beta) \cdot \gamma$.

Proof. □

Proposition 1.3.6: Multiplicative identity

$\alpha \cdot \mathbf{1} = \alpha = \mathbf{1} \cdot \alpha$.

Proof. □

Lemma 1.3.7

If $q \in \mathbb{Q}$ and $q > 1$, then set $S = \{q^n : n \in \mathbb{N}\}$ is unbounded from above, namely for any $M \in \mathbb{Q}$, there is $n \in \mathbb{N}$ such that $q^n > M$.

Proof. Exercise. *Hint:* It suffices to prove by induction that $q^n \geq 1 + n(q - 1)$ for all $n \in \mathbb{N}$. □

Lemma 1.3.8

If $q \in \mathbb{Q}$ and $0 < q < 1$, then $\alpha \cap \{q^n : n \in \mathbb{N}\} \neq \varnothing$ for any real number $\alpha > \mathbf{0}$.

Proof. Exercise. *Hint:* Use the fact that $\frac{1}{q} > 1$ and the result in the hint given in Lemma 2. □

Definition 1.3.9: Multiplicative Inverses

For $\alpha > 0$, we define the set α^{-1} by

$$\alpha^{-1} = \left\{x \in \mathbb{Q}_{>0} : \tfrac{1}{x} \in \mathbb{Q} \setminus \alpha \text{ and } \tfrac{1}{x} \text{ is not the minimum of } \mathbb{Q} \setminus \alpha\right\} \cup \mathbb{Q}_{\leq 0}$$

If $\alpha < \mathbf{0}$, we define $\alpha^{-1} = -(|\alpha|)^{-1} = -((-\alpha)^{-1})$.

As one would expect, α^{-1} should satisfy $\alpha^{-1}\alpha = \mathbf{1}$. This won't follow directly from the definition, but it is something that we can prove. Before we can do so, we need to verify that α^{-1} is actually a real number first.

Proposition 1.3.10: Closure

If $\alpha \neq \mathbf{0}$, then α^{-1} is a real number.

Proof. Again, it suffices to prove this in the case that $\alpha > \mathbf{0}$. Clearly $-1 \in \alpha^{-1}$, so $\alpha^{-1} \neq \varnothing$. Similarly, as $\alpha > 0$, we can find a positive rational number $x \in \alpha$. Then $\frac{1}{x} \notin \alpha^{-1}$ as $1/\left(\frac{1}{x}\right) = x \in \alpha$, and so $\alpha^{-1} \neq \mathbb{Q}$. Now let $z \in \alpha^{-1}, q \in \mathbb{Q}$ and suppose $q < z$. If $q \leq 0$, then $q \in \alpha^{-1}$ immediately. If $q > 0$, then $0 < q < x$ and so $0 < \frac{1}{x} < \frac{1}{q}$ and since $\frac{1}{x} \in \mathbb{Q} \setminus \alpha$, we must have $\frac{1}{q} \in \mathbb{Q} \setminus \alpha$ and thus $q \in \alpha^{-1}$. Finally, we show α^{-1} has no maximum, where we will implicitly show that $\alpha^{-1} > \mathbf{0}$. Let $x \in \alpha^{-1}$ and suppose $x \leq 0$. Since $\alpha > \mathbf{0}$ and $\alpha \neq \mathbb{Q}$, there is $a \in \mathbb{Q}_{>0}$ such that $a \in \mathbb{Q} \setminus \alpha$, moreover we may assume a is not the minimum of $\mathbb{Q} \setminus \alpha$ by taking any rational number larger than a. Then $\frac{1}{a} \in \alpha^{-1}$, and so $x \leq 0 < \frac{1}{a}$. If $x > 0$, then we have $\frac{1}{x} \in \mathbb{Q} \setminus \alpha$ and there is some $y \in \mathbb{Q} \setminus \alpha$ such that $y < \frac{1}{x}$ as $\frac{1}{x}$ is not the minimum of $\mathbb{Q} \setminus \alpha$. Choose some $q \in \mathbb{Q}$ such that $y < q < \frac{1}{x}$, then $\frac{1}{q} \in \alpha^{-1}$ as $q > y \Rightarrow \notin \alpha$ and $x < \frac{1}{q}$. Thus α^{-1} is a real number. □

Proposition 1.3.11: Multiplicative inverses

If $\alpha \neq \mathbf{0}$, then $\alpha \cdot \alpha^{-1} = \mathbf{1}$.

Proof. □

1.4 Dedekind's Theorem

1.5 Consequences of Completeness

Theorem 1.5.1: Criterion for Supremum

Let $A \subseteq \mathbb{R}$ be a non-empty set, bounded above. For any upper bound $M \in \mathbb{R}$ of A, $M = \sup(A)$ if and only if for every $\varepsilon > 0$, there is $a \in A$ such that

$$M - \varepsilon < a \leq M$$

Proof. ($\Rightarrow$) By the minimality of $M = \sup(A)$, $M - \varepsilon$ is not an upper bound for any $\varepsilon > 0$, and hence there is $a \in A$ such that $\alpha - \varepsilon < a$.

($\Leftarrow$) As M is an upper bound, we have $\sup(A) \leq M$, suppose towards a contradiction $\sup(A) < M$. By assumption there is $a \in A$ such that $a > M - (M - \sup(A)) = \sup(A)$, which is a contradiction. □

Proposition 1.5.2: Completeness for Infima

Let $A \subseteq \mathbb{R}$ be a non-empty set, bounded below. Then

$$-\sup(\{-a : a \in A\})$$

is the greatest lower bound for A.

2 Limits and Continuity

Limits are likely a familiar notion from high school calculus, they help us describe the behaviour of a function in a small area around a particular point. Limits are the building blocks of Calculus and Analysis, all of the important concepts we will cover – Continuity, Differentiability, Integrability, Sequences and Series – are all based around limits. With that in mind, it is imperative that you are able to grasp the concepts outlined in this section, as without them the rest of the course will surely be an uphill battle.

2.1 Limits of Functions

Let's begin in (hopefully) a familiar territory – limits of functions. This was likely covered in high school calculus, but here we are looking to take a more formal approach to the idea of a limit. Our formal definition of a limit will be a tool that we will use very frequently throughout the section and the rest of the notes.

Definition 2.1.1: Formal Limits

Let $a \in \mathbb{R}$ and $f : U \to \mathbb{R}$ a function defined in a neighbourhood of a, except possibly at a. We say that *the limit of f as x approaches a is L*, written $\lim_{x\to a} f(x) = L$, if for every $\varepsilon > 0$, there is $\delta > 0$ such that for all $x \in U$, if $0 < |x - a| < \delta$, then $|f(x) - L| < \varepsilon$.

The formal definition of a limit will probably seem very scary and complicated at first, and that's ok. Let's take a step back and look at things informally. To start, it is imperative that you view an inequality of the form $|f(x) - L| < \varepsilon$ as saying that "the **distance** between $f(x)$ and L is less than ε." namely that the absolute value of the difference should be understood to be a distance. It's also important to view $\varepsilon > 0$ as a pre-specified error term, namely $\varepsilon > 0$ represents the error on our approximation of the value L by using values of the form $f(x)$ for x close to a. From here, the definition reads as follows:

Given an error bound $\varepsilon > 0$, we can find $\delta > 0$ so that for any x in the domain of our function, if the distance between x and a is smaller than δ and $x \neq a$, then we can guarantee the distance between $f(x)$ and L is smaller than our error bound $\varepsilon > 0$

We're going to be working with this definition quite a lot, so let's spend some time going through several examples to get our hands dirty.

Example 2.1.2: An Illustration

Let $a, L \in \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ be a function. Suppose that there is some $K > 0$ such that $|f(x) - L| \leq K|x - a|$ for all $x \in \mathbb{R}$. Prove that $\lim_{x\to a} f(x) = L$.

Solution. The purpose of this example is to illustrate a general approach for solving problems and writing proofs involving limits. Let $\varepsilon > 0$. The key is to note that if $K|x - a| < \varepsilon$, then follows that

$$|f(x) - L| \leq K|x - a| < \varepsilon$$

Note that $K|x - a| < \varepsilon$ is equivalent to $|x - a| < \frac{\varepsilon}{K}$. Recall from Definition 2.1.1 that δ is allowed to depend on ε, and so a good choice for δ would be $\delta = \frac{\varepsilon}{K} > 0$. To verify this, let $x \in \mathbb{R}$ and assume that $0 < |x-a| < \delta$. Then, it follows that

$$|f(x) - L| \leq K|x-a| < K\delta = K\frac{\varepsilon}{K} = \varepsilon$$

Thus, by Definition 2.1.1, we conclude that $\lim_{x\to a} f(x) = L$. □

Annecdote about doing rough work and generalizing the previous example

Example 2.1.3

Prove that $\lim_{x\to 4}(3x+7) = 19$.

Proof. □

Example 2.1.4: Limits of Linear Functions

Let $a, b, m \in \mathbb{R}$ with $m \neq 0$. Prove that $\lim_{x\to a}(mx+b) = ma+b$.

Proof. Let $\varepsilon > 0$ and pick $\delta = \frac{\varepsilon}{|m|} > 0$. Given $x \in \mathbb{R}$, it follows that

$$0 < |x-a| < \delta \quad \Longrightarrow \quad |(mx+b) - (ma+b)| = |m||x-a| < |m|\delta = \varepsilon$$ □

Example 2.1.5

Prove that $\lim_{x\to 2}(x^2 - x + 7) = 9$.

Proof. □

Example 2.1.6

Prove that $\lim_{x\to 1}\left(\frac{x+5}{x+2}\right) = 2$.

Proof. □

Example 2.1.7

Prove that $\lim_{x\to 4}\sqrt{x+5} = 3$.

Proof. □

Example 2.1.8: A Non-Example

Define the piecewise function $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} x & x \neq -3 \\ 0 & x = -3 \end{cases}$$

Prove that $\lim_{x\to -3} f(x) \neq 0$.

Proof. □

Example 2.1.9: Non-Existent Limit

Consider the function $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ defined by $f(x) = \frac{x}{|x|}$. Prove that $\lim_{x\to 0} f(x)$ does not exist.

Proof. □

2.2 Types of Limits

So far, we have discussed the local behaviour of a function in a two-sided neighbourhood of a point in its domain. There may be times in our study that we are only interested in the behaviour of our function on one of the two sides of a given point. For example, suppose $f : [0, \infty) \to \mathbb{R}$ is defined by $f(x) = \sqrt{x}$. We cannot study the behaviour of the function for any values $a < 0$ since f is not defined there. So to investigate the behaviour of f near 0, we would be doing so *to the right of* 0. We can formalize this idea of a one-sided limit to get the following definition.

Definition 2.2.1: One-Sided Limits

Let $a \in \mathbb{R}$ and suppose $f : (a, a + \rho_1) \to \mathbb{R}$ and $g : (a - \rho_2, a) \to \mathbb{R}$ are functions, for some $\rho_1, \rho_2 > 0$. We say that

1. *The limit of f as x approaches a from the right is L_1*, written $\lim_{x\to a^+} f(x) = L_1$, if for all $\varepsilon > 0$, there is $\delta > 0$ such that for all $x \in (a, a+\rho_1)$; if $x \in (a, a+\delta)$ then $|f(x) - L_1| < \varepsilon$.
2. *The limit of g as x approaches a from the left is L_2*, written $\lim_{x\to a^-} g(x) = L_2$, if for all $\varepsilon > 0$, there is $\delta > 0$ such that for all $x \in (a - \rho_2, a)$; if $x \in (a - \delta, a)$ then $|g(x) - L_2| < \varepsilon$.

Recall that we can write $|x - a| < \delta$ as $x \in (a - \delta, a + \delta)$. Thus, if we also impose that $0 < |x - a|$, namely $x \neq a$, we can say that

$$0 < |x - a| < \delta \quad \iff \quad x \in (a - \delta, a) \cup (a, a + \delta)$$

This is precisely what we mean when we say that one sided limits capture the behaviour of a function one one side of a particular point, either the right (values larger) or the left (values smaller).

Example 2.2.2

Prove that $\lim_{x\to 4^+} x\sqrt{x - 4} = 0$.

Proof. □

One thing you may be wondering is how we can infer information about a two-sided limit using one sided-limits. It should hopefully be intuitive that if a two-sided limit exists, then so too should the one sided limits and moreover their values should all be the same. The converse however is also true, as demonstrated by the following important result.

Proposition 2.2.3: Criterion for Two-sided Limits

Let $a \in \mathbb{R}$ and $f : U \to \mathbb{R}$ a function defined on a neighbourhood of a, except possibly at a. Then

$$\lim_{x\to a} f(x) = L \qquad \Longleftrightarrow \qquad \lim_{x\to a^-} f(x) = L = \lim_{x\to a^+} f(x)$$

Proof. □

The remaining types of limits we will explore in this section deal with infinity. Namely when we are interested in the behaviour of our function as x approaches either positive or negative infinity, as well as what it means for the limit, as x approaches a finite value, of a function to be either positive or negative infinity. The definitions will mostly be the same as what we've seen, however we will have to change either

$$0 < |x - a| < \delta \quad \text{or} \quad |f(x) - L| < \varepsilon$$

to account for infinities. Intuitively, a limit should equal positive (negative) infinity if we can always make the value of the function larger (smaller) than any real number. Let's make this precise

Definition 2.2.4: Infinite Limits and Vertical Asymptotes

Let $a \in \mathbb{R}$ and $f : U \to \mathbb{R}$ a function defined on a neighbourhood of a, except possibly at a. We say that *the limit of f as x approaches a is positive (negative) infinity*, written $\lim_{x\to a} f(x) = \infty$ ($\lim_{x\to a} f(x) = -\infty$) if for all $M \in \mathbb{R}$, there is $\delta > 0$ such that for all $x \in U$; if $0 < |x - a| < \delta$ then $f(x) > N$ ($f(x) < N$).

We say that f has a *vertical asymptote at a* if either $\lim_{x\to a^+} f(x) = \pm\infty$ or $\lim_{x\to a^-} f(x) = \pm\infty$.

Example 2.2.5

Prove that $\lim_{x\to 0^-} \frac{1}{x^2} = 0$.

Proof. □

Example 2.2.6

Prove that $\lim_{x\to 1^-} \frac{x}{x^2-1} = \infty$.

Proof. □

Analogously, if we want to formalize what it means for a function to approach a finite value L as $x \to \pm\infty$, we can do so as follows

Definition 2.2.7: Limits at Infinity and Horizontal Asymptotes

Let $a \in \mathbb{R}$ and suppose $f : (a, \infty) \to \mathbb{R}, g : (-\infty, a) \to \mathbb{R}$ are functions. We say that

1. *The limit of f as x approaches positive infinity is L_1*, written $\lim_{x\to\infty} f(x) = L_1$, if for all $\varepsilon > 0$, there is $N \in \mathbb{R}$ such that for all $x \in (a, a+\infty)$; if $x > N$ then $|f(x) - L_1| < \varepsilon$.
2. *The limit of g as x approaches negative infinity is L_2*, written $\lim_{x\to\infty} g(x) = L_2$, if for all $\varepsilon > 0$, there is $N \in \mathbb{R}$ such that for all $x \in (a-\infty, a)$; if $x < N$ then $|g(x) - L_2| < \varepsilon$.

In such cases, we say that f and g have *horizontal asymptotes* of L_1 and L_2.

Example 2.2.8

Prove that $\lim_{x\to\infty} \left(\frac{6x-2}{3x+7}\right) = 2$

Proof. □

There is a nice relationship we can formulate between limits at infinity and one-sided limits. It hinders on the observation that if x is very small and positive, then $\frac{1}{x}$ is very large, and vice versa (analogously for the negative case). This leads to the following transformation rule.

Proposition 2.2.9: Transformation I

Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Then

1. $\lim_{x\to\infty} f\left(\frac{1}{x}\right) = L$ if and only if $\lim_{x\to 0^+} f(x) = L$.
2. $\lim_{x\to-\infty} f\left(\frac{1}{x}\right) = L$ if and only if $\lim_{x\to 0^-} f(x) = L$.

comments on the converse and generalizations

2.3 Properties of Limits

In this section, our goal is to make less work for ourselves later on. Namely, we can derive some useful properties of limits so that we don't have to resort to using the definition every time. To start, we can address an issue of well-definedness, namely that the limit of a function is unique (assuming that it exists) and that we are correct in saying *the* limit.

Proposition 2.3.1: Uniqueness of Limits

Suppose $a \in \mathbb{R}$ and $f : U \to \mathbb{R}$ is a function defined on a neighbourhood of a, except possibly at a. If $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} f(x) = M$, then $L = M$.

Proof. □

From here, we can get into what is likely one of the main theorems that you learned in high school, the limit laws. This result will allow us to compute new limits from old ones, and will save us a lot of work later on! For now, there are a few small results we will need ahead of time before we can actually prove the limits laws.

Lemma 2.3.2: Local Boundedness

Let $a \in \mathbb{R}$ and $f : U \to \mathbb{R}$ a function defined in a neighbourhood of a, except possibly at a. If $\lim_{x\to a} f(x) = L$, then there is $B, D > 0$ and $\delta_1, \delta_2 > 0$ such that

1. $|f(x)| \leq B$ for all x such that $0 < |x - a| < \delta_1$.
2. $|f(x)| \geq D$ for all x such that $0 < |x - a| < \delta_2$, assuming that $L \neq 0$.

Proof. □

Now that we have everything we need, we can jump into proving the limit laws.

discussion about the $\frac{\varepsilon}{2}$ trick and the $\min\{\delta_1, \delta_2\}$ trick.

Theorem 2.3.3: The Limit Laws

Let $a \in \mathbb{R}, f, g : U \to \mathbb{R}$ functions defined in a neighbourhood of a, except possibly at a, and assume that $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$. Then

1. $\lim_{x\to a}[f(x) + g(x)] = L + M$.
2. $\lim_{x\to a} \alpha f(x) = \alpha L$ for all $\alpha \in \mathbb{R}$.
3. $\lim_{x\to a} f(x)g(x) = LM$.
4. $\lim_{x\to a} \frac{1}{f(x)} = \frac{1}{L}$, assuming $L \neq 0$.

Proof. □

Another familiar result from high school calculus is likely that of the squeeze theorem. Here we will state and prove a stronger version of it by only assuming that the inequalities hold locally rather than globally.

Theorem 2.3.4: Squeeze Theorem

Let $a \in \mathbb{R}$ and $f, g, h : U \to \mathbb{R}$ be functions defined in a neighbourhood of a, except possibly at a. If $\lim_{x\to a} f(x) = L = \lim_{x\to a} h(x)$ and there is $\delta > 0$ such that for all $0 < |x - a| < \delta$ we have

$$f(x) \leq g(x) \leq h(x)$$

Then $\lim_{x\to a} g(x) = L$.

Proof. □

Proposition 2.3.5: Order and Limits

Let $a \in \mathbb{R}, \rho > 0$ and $f, g : (a-\rho, a) \cup (a, a+\rho) \to \mathbb{R}$ functions for which $\lim_{x\to a} f(x), \lim_{x\to a} g(x)$ both exist and $f(x) \leq g(x)$ for all $x \in (a - \rho, a) \cup (a, a + \rho)$. Then

$$\lim_{x\to a} f(x) \leq \lim_{x\to a} g(x)$$

Proof. □

Theorem 2.3.6: Monotone Convergence

Let $a \in \mathbb{R}$ and suppose $f : [a, \infty) \to \mathbb{R}$ is a bounded, increasing function. Then

$$\lim_{x\to\infty} f(x) = \sup_{x\geq a} f(x)$$

Proof. Let $\varepsilon > 0$ and define the set $A = \{f(x) : x \geq a\}$. As f is bounded, A is bounded by Definition **??** and A is non-empty as $f(a) \in A$ by construction. Thus, by completeness, $\sup(A)$ exists. By Theorem **??**, there is $x_0 \geq a$ for which

$$\sup(A) - \varepsilon < f(x_0) \leq \sup(A) < \sup(A) + \varepsilon$$

Finally, by the monotonicity of f, for any $x \in \mathbb{R}$ such that $x > x_0$, we have

$$\sup(A) - \varepsilon < \underbrace{f(x_0) \leq f(x)}_{\text{monotonicity}} \leq \sup(A) < \sup(A) + \varepsilon \quad \iff \quad |f(x) - \sup(A)| < \varepsilon$$ □

Theorem 2.3.7: Monotone One-sided Limits

Let $f : \mathbb{R} \to \mathbb{R}$ be a monotone function. Then for every $a \in \mathbb{R}$

$$\lim_{x\to a^+} f(x) = \inf_{x>a} f(x) \quad \text{and} \quad \lim_{x\to a^-} f(x) = \sup_{x<a} f(x)$$

Proof. This is very similar to the proof of the monotone convergence theorem. Define the sets

$$A^- = \{f(x) : x < a\} \quad \text{and} \quad A^+ = \{f(x) : x > a\}$$

By our assumption that f is monotone increasing, we have that A^- is bounded above and A^+ is bounded below, both by $f(a)$, so $\sup(A^-)$ and $\inf(A^+)$ both exist. We start by showing the left hand limit at a is $\sup(A^-)$. Let $\varepsilon > 0$, by Theorem **??** there is $y \in A^-$ such that $\sup(A^-) - \varepsilon < y$, namely writing $y = f(x_0)$ for some $x_0 < a$, we have $\sup(A^-) - \varepsilon < f(x_0)$. Let $\delta = a - x_0 > 0$, then for any $x \in (a - \delta, a)$, we have $x_0 < x < a$, and so as f is strictly increasing

$$\sup(A^-) - \varepsilon < f(x_0) < f(x) \leq \sup(A^-) < \sup(A^-) + \varepsilon$$

Thus, $\sup(A^-) - \varepsilon < f(x) < \sup(A^-) + \varepsilon$, namely $|f(x) - \sup(A^-)| < \varepsilon$. A similar argument shows that the right hand limit is $\inf(A^+)$. □

2.4 Continuity and Uniform Continuity

In this section we will introduce one of the most important concepts in the course, that of continuity and continuous functions. Continuity is likely not an unfamiliar concept, however, the mantra that "a function is continuous if you can draw its graph without lifting your pencil off of the paper" simply will not do. Luckily for us, the formal definition of continuity is very similar to our definition of a limit, with some notable differences.

Definition 2.4.1: Continuity

Let $a \in \mathbb{R}$ and $f : U \to \mathbb{R}$ a function defined in a neighbourhood of a. We say that f *is continuous at* a if $\lim_{x\to a} f(x) = f(a)$. We say that f *is continuous* if f is continuous at a for all $a \in U$. Finally, if $U = \mathbb{R}$ and f is continuous, we will say that f *is everywhere continuous*, namely if f is continuous at a for all $a \in \mathbb{R}$.

Note that in order for a function to be continuous at a point, it must be defined at the point in question. By using our definition of a limit, we can say that f is continuous at a if

$$\text{for all } \varepsilon > 0, \text{ there is } \delta > 0 \text{ such that for all } x \in U; \text{ if } |x-a| < \delta \text{ then } |f(x) - f(a)| < \varepsilon.$$

Note that this is essentially our original limit definition with $L = f(a)$. The only other difference is that we don't require $0 < |x-a|$. Since f is defined at a, clearly $|a-a| = 0 < \delta$ for any $\delta > 0$, and moreover $|f(a) - f(a)| = 0 < \varepsilon$ for any $\varepsilon > 0$. Thus, we do not need to enforce that $x \neq a$.

Example 2.4.2

Prove that each of the following functions are continuous.

1. $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = |x|$.
2. $g : [0, \infty) \to \mathbb{R}$ defined by $g(x) = \sqrt{x}$.

Proof. □

comments about pathological functions and the need for a rigorous definition

Example 2.4.3: A Nowhere Continuous Function

We define the *characteristic function of the rationals*, or *Dirichlet's function* $D : \mathbb{R} \to \mathbb{R}$ by

$$D(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

Prove that D is continuous nowhere, namely D is discontinuous at a for all $a \in \mathbb{R}$.

Proof. □

Example 2.4.4: Thomae's Function

We define *Thomae's Function* $T : \mathbb{R} \to \mathbb{R}$ as follows

$$T(x) = \begin{cases} \frac{1}{q} & x = \frac{p}{q} \in \mathbb{Q}, p \in \mathbb{Z}, q \in \mathbb{N}, \gcd(p,q) = 1 \\ 0 & x \in \mathbb{Q}^c \cup \{0\} \end{cases}$$

Prove that $\lim_{x \to a} T(x) = 0$ for all $a \in \mathbb{R}$ and conclude on the continuity of Thomae's function.

Proof. □

Proposition 2.4.5: Change of Variables

Let $a, L \in \mathbb{R}$ and $g : U \to \mathbb{R}, f : V \to \mathbb{R}$ functions defined on neighbourhoods U of a and V of L, except possibly at the respective points, such that $\lim_{x\to a} g(x) = L$ and $\lim_{x\to L} f(x) = M$. If either of the conditions:

1. f is defined at L and $M = f(L)$ (namely f is continuous at L).
2. There is $\rho > 0$ for which $g(x) \neq L$ for all $x \in (a - \rho, a) \cup (a, a + \rho)$.

hold, then $\lim_{x\to a}(f \circ g)(x) = M$.

Proof. □

Corollary 2.4.6: Continuity and Composition

Let $a \in \mathbb{R}$ and $g : U \to \mathbb{R}, f : V \to \mathbb{R}$ be functions continuous at a and $g(a)$ respectively, where $g(U) \subseteq V$. Then $f \circ g : U \to \mathbb{R}$ is continuous at a.

Proof. □

Definition 2.4.7: Types of Discontinuities

Definition 2.4.8: Uniform Continuity

Consider $D \subseteq \mathbb{R}$ and a function $f : D \to \mathbb{R}$. We say that f is *uniformly continuous* if for all $\varepsilon > 0$, there is $\delta > 0$ such that for all $x, y \in D$, if $|x - y| < \delta$ then $|f(x) - f(y)| < \varepsilon$.

comments about the definition, distinction from regular pointwise continuity.

Example 2.4.9

Prove that each of the following functions are uniformly continuous.

1. $f : [1, \infty) \to \mathbb{R}$ defined by $f(x) = \frac{1}{x}$.
2. $g : [0, \infty) \to \mathbb{R}$ defined by $g(x) = \sqrt{x}$.

Proof. (1) Let $\varepsilon > 0$ be given and choose $\delta = \varepsilon$. Let $x, y \in [1, \infty)$ be given, Note that $x, y \geq 1$ so $xy \geq 1$ and moreover

$$|f(x) - f(y)| = \left|\frac{1}{x} - \frac{1}{y}\right| = \frac{|x - y|}{xy} \leq |x - y| < \delta < \varepsilon$$

(2) Let $\varepsilon > 0$ be given, and choose δ □

comments on the above example, set up for the proceeding theorem.

Theorem 2.4.10: Criterion for Uniform Continuity

If $f : [a, b] \to \mathbb{R}$ is continuous, then f is uniformly continuous.

Proof. Let $\varepsilon > 0$, and define the following set

$$U = \{c \in [a,b] : \exists \delta > 0 \text{ such that } \forall x, y \in [a,c], |x-y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon\}$$

U is the collection of points $c \in [a,b]$ for which f is uniformly continuous on $[a,c]$ given $\varepsilon > 0$. Our goal will be to show $U = [a,b]$. Note that $a \in U$ as $[a,a] = \{a\}$, and so any $\delta > 0$ will work. Also, $U \subseteq [a,b]$ by construction, so $c \leq b$ for all $c \in U$. Thus, $\alpha := \sup(U)$ exists by completeness, and $\alpha \leq b$. We want to show $U = [a,b]$, so we need to show $b \in U$. Suppose $\alpha < b$, as f is continuous at α, there is $\delta_1 > 0$ such that for all $x \in [a,b], |x - \alpha| < \delta_1 \Rightarrow |f(x) - f(\alpha)| < \frac{\varepsilon}{2}$. By the criterion for supremum, there is $c \in U$ such that $\alpha - \delta_1 < c \leq \alpha$, and as $c \in U$, there is $\delta_2 > 0$ such that for all $x, y \in [a,c], |x-y| < \delta_2 \Rightarrow |f(x) - f(y)| < \varepsilon$. Set $\delta = \min\left\{\frac{\delta_1}{2}, \delta_2\right\}$, to arrive at a contradiction, we show $\alpha + \frac{\delta}{2} \in U$. Fix $x, y \in \left[a, \alpha + \frac{\delta}{2}\right]$, and suppose $|x - y| < \delta$. If $x, y \in \left[a, \alpha - \frac{\delta_1}{2}\right]$, then $x, y \in [a,c]$, and so as $c \in U, |x-y| < \delta \leq \delta_2 \Rightarrow |f(x) - f(y)| < \varepsilon$. Conversely, without loss of generality if $x \in \left(\alpha - \frac{\delta_1}{2}, \alpha + \frac{\delta}{2}\right]$, then $|x - \alpha| < \frac{\delta_1}{2}$, and as $|x - y| < \delta$, it follows by the triangle inequality that $|y - \alpha| \leq |x - y| + |x - \alpha| < \delta + \frac{\delta_1}{2} \leq \frac{\delta_1}{2} + \frac{\delta_1}{2} = \delta_1$, and so

$$|f(x) - f(y)| \leq |f(x) - f(\alpha)| + |f(y) - f(\alpha)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Thus, $\alpha + \frac{\delta}{2} \in U$, contradicting that $\alpha = \sup(U)$, and so $\alpha = b$. Lastly, we show $b = \alpha \in U$. By the left continuity of f at b, there is $\delta_1 > 0$ such that $x \in (b - \delta_1, b] \Rightarrow |f(x) - f(y)| < \frac{\varepsilon}{2}$. As $b = \sup(U)$, there is $c \in U$ such that $c \in (b - \delta_1, b]$, and so there is $\delta_2 > 0$ such that for all $x, y \in [a,c], |x-y| < \delta_2 \Rightarrow |f(x) - f(y)| < \varepsilon$. Set $\delta = \min\left\{\frac{\delta_1}{2}, \delta_2\right\}$ and suppose $x, y \in [a,b]$ such that $|x - y| < \delta$. If $x, y \in \left[a, b - \frac{\delta_1}{2}\right] \subseteq [a,c]$, then $|x - y| < \delta \leq \delta_2 \Rightarrow |f(x) - f(y)| < \varepsilon$. If, without loss of generality, $x \in \left(b - \frac{\delta_1}{2}, b\right]$, then $|x - b| < \frac{\delta_1}{2}$, and by the triangle inequality, $|y - b| \leq |x - y| + |x - b| < \frac{\delta_1}{2} + \delta \leq 2\frac{\delta_1}{2} = \delta_1$, and so it follows that

$$|f(x) - f(y)| \leq |f(x) - f(b)| + |f(y) - f(b)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Thus, $b \in U$ and hence f is uniformly continuous. □

It's worth noting what goes wrong when we remove the conditions that the domain is both closed and bounded, as seen in the following examples.

Example 2.4.11: Counter Example I

Show that the function $f : (0,1) \to \mathbb{R}, f(x) = \frac{1}{x}$ is continuous but not uniformly continuous.

Proof. Fix $c \in (0,1)$, it follows from the limit laws that f is continuous at c, and so f is continuous on $(0,1)$. We claim f is *not* uniformly continuous. Indeed, let $\varepsilon_0 = \frac{1}{2}$ and $\delta > 0$ be given. Choose some $x_0 \in (0,1)$ such that $0 < x_0 < \min\{1, \delta\}$ and set $y_0 = \frac{x_0}{2}$. Note that $|x_0 - y_0| = \frac{x_0}{2} < \delta$ and

$$|f(x_0) - f(y_0)| = \left|\frac{1}{x_0} - \frac{1}{y_0}\right| = \frac{|x_0 - y_0|}{x_0 y_0} = \frac{x_0}{2}\left(\frac{1}{x_0 y_0}\right) = \frac{1}{2y_0} \geq \frac{1}{2} = \varepsilon_0$$ □

Example 2.4.12: Counter Example II

Show that $f : (0,\infty) \to \mathbb{R}, f(x) = x^2$ is continuous but not uniformly continuous.

Proof. Let $c \in [0, \infty)$, it follows from the limit laws that f is continuous at c at hence continuous on $[0, \infty)$, we show f is not uniformly continuous. Pick $\varepsilon_0 = 1$ and let $\delta > 0$ be given. Set $x_0 = \frac{1}{\delta}$ and $y_0 = x_0 + \frac{\delta}{2}$. Then $|x_0 - y_0| = \frac{\delta}{2} < \delta$, and

$$|f(x_0) - f(y_0)| = x_0\delta + \frac{\delta^2}{4} > x_0\delta = 1 = \varepsilon_0$$

Thus, f is not uniformly continuous. $\square$

This result and Example 2.4.11 Show that the assumption of Theorem 2.4.12 above fails if we no longer assume that the function is continuous on a closed and bounded interval.

2.5 The Intermediate Value Theorem and Friends

2.6 Big and Little O Notation

3 Integration

3.1 The Darboux Integral

Definition 3.1.1: Partitions

Let $[a,b]$ be an interval. An ordered collection of points $P = \{x_0, \ldots, x_n\}$ is called a partition of $[a,b]$ if each $x_i \in [a,b]$ and $x_0 = a, x_n = b$. Given an index $i \in \{1, \ldots, n\}$ we say that $[x_{i-1}, x_i]$ is the i^{th} subinterval of P and define $\Delta x_i = x_i - x_{i-1}$ to be its length. We define the norm of a partition $P = \{x_0, \ldots, x_n\}$, denoted $\|P\|$, to be the length of its longest subinterval, namely

$$\|P\| = \max_{1 \leq i \leq n} \Delta x_i$$

We denote the collection of partitions of $[a,b]$ as $\mathcal{P}_{[a,b]}$. Given $P, Q \in \mathcal{P}_{[a,b]}$, we say Q is a refinement of P, or Q refines P, if $P \subseteq Q$. Finally, we say that $P \cup Q$ is the common refinement of P and Q.

Definition 3.1.2: Upper and Lower Darboux Sums

Given a bounded function $f : [a,b] \to \mathbb{R}$ and a partition $P = \{x_i\}_{i=1}^n \in \mathcal{P}_{[a,b]}$, we define the upper and lower Darboux sums of f on P, denoted $U(f,P)$ and $L(f,P)$ respectively, by

$$U(f,P) = \sum_{i=1}^n \left[\sup_{x \in [x_{i-1}, x_i]} f(x) \right] \Delta x_i \quad \text{and} \quad L(f,P) = \sum_{i=1}^n \left[\inf_{x \in [x_{i-1}, x_i]} f(x) \right] \Delta x_i$$

Additionally, we define the upper and lower Darboux integrals of f by

$$U(f) = \inf_{P \in \mathcal{P}_{[a,b]}} U(f,P) \quad \text{and} \quad L(f) = \sup_{P \in \mathcal{P}_{[a,b]}} L(f,P)$$

Lemma 3.1.3: Properties of Partitions

Let P and Q be partitions of an interval $[a,b]$. The following hold

1. $L(f,P) \leq U(f,P)$.
2. If P is a refinement of Q, namely $P \subseteq Q$, then
$$L(f,P) \leq L(f,Q) \leq U(f,Q) \leq U(f,P)$$
3. $L(f,P) \leq U(f,Q)$.

Proof. Let's write $P = \{x_0, \ldots, x_n\}$ where $x_0 = a$ and $x_n = b$.

1. This follows immediately as $\inf_{x \in [x_{i-1}, x_i]} f(x) \leq \sup_{x \in [x_{i-1}, x_i]} f(x)$ for any subinterval.

2. It suffices to prove this in the case where Q refines P by a single point, namely $Q = P \cup \{c\}$ where $c \notin P$. As $c \notin P$, there is an index $j \in \{1, \ldots, n\}$ for which $c \in [x_{j-1}, x_j]$. Note that as

$[x_{j-1}, c], [c, x_j] \subseteq [x_{j-1}, x_j]$, we have by properties of supremums and infimums that

$$\begin{aligned}\sup_{x\in[x_{j-1},x_j]} f(x)\Delta x_j &= \sup_{x\in[x_{j-1},x_j]} f(x)(x_j - c) + \sup_{x\in[x_{j-1},x_j]} f(x)(c - x_{j-1}) \\ &\geq \sup_{x\in[x_{j-1},c]} f(x)(x_j - c) + \sup_{x\in[c,x_j]} f(x)(c - x_{j-1})\end{aligned}$$

$$\begin{aligned}\inf_{x\in[x_{j-1},x_j]} f(x)\Delta x_j &= \inf_{x\in[x_{j-1},x_j]} f(x)(x_j - c) + \inf_{x\in[x_{j-1},x_j]} f(x)(c - x_{j-1}) \\ &\leq \inf_{x\in[x_{j-1},c]} f(x)(x_j - c) + \inf_{x\in[c,x_j]} f(x)(c - x_{j-1})\end{aligned}$$

The result follows as P and Q are identical on all other subintervals $[x_{i-1}, x_i]$ for $i \neq j$.

3. By considering the common refinement $P \cup Q$ of P and Q, we can apply 2 and the result follows. □

Definition 3.1.4: Darboux Integrability

A bounded function $f : [a, b] \to \mathbb{R}$ is called Darboux integrable if $L(f) = U(f)$, and the Darboux integral of f is defined to be their common value.

This definition should hopefully seem natural, in general, we have that $U(f, P)$ is an over approximation of the integral and $L(f, P)$ and under approximation. We would want the integral to be the value that our over and under approximations "converge" to. Note that this same thought motivates why statement 2 in Lemma 3.1.3 is true, when we add more points to our partition, our over and under approximations become more accurate.

Example 3.1.5: Constant Functions

If $f : [a, b] \to \mathbb{R}$ is constant, then f is integrable.

Proof. □

Lemma 3.1.6: Integrability Criterion

A function $f : [a, b] \to \mathbb{R}$ is Darboux integrable if and only if for every $\varepsilon > 0$, there is a partition $P \in \mathcal{P}_{[a,b]}$ such that $U(f, P) - L(f, P) < \varepsilon$. Moreover, the Darboux integral I is the unique value such that $L(f, P) \leq I \leq U(f, P)$ for all $P \in \mathcal{P}_{[a,b]}$.

This should also feel relatively natural. Given a partition P, the quantity $U(f, P) - L(f, P)$ represents the "error" of our approximations using the partition P. Thus, if we can always choose a partition to make our error arbitrarily small, it follows that our over and under approximations converge.

Example 3.1.7

Piece-wise function

Now that we've seen some examples of integrable functions, we can see an example of a function that is *not* integrable. As per Lemma 3.1.6, to show that a bounded function $f : [a, b] \to \mathbb{R}$ is *not* integrable, we show that there is $\varepsilon_0 > 0$ such that $U(f, P) - L(f, P) \geq \varepsilon_0$ for all partitions P of $[a, b]$

Example 3.1.8: A Non-Integrable Function

3.2 Properties of The Darboux Integral

Definition 3.2.1: Oscillation of a Function

Let $f : [a, b] \to \mathbb{R}$ be a bounded function and $[c, d] \subseteq [a, b]$. We define the oscillation of f on $[c, d]$, denoted $\omega(f, [c, d])$, by

$$\omega(f, [c, d]) = \sup_{x,y \in [c,d]} |f(x) - f(y)|$$

Lemma 3.2.2: Properties of Oscillation

Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Then for any subinterval $[c, d] \subseteq [a, b]$ we have

1. $\omega(f, [c, d]) = \sup_{x\in[c,d]} f(x) - \inf_{x\in[c,d]} f(x)$.
2. $\omega(|f|, [c, d]) \leq \omega(f, [c, d])$.
3. For any partition $P = \{x_0, \ldots, x_n\}$ of $[a, b]$, we can write

$$U(f, P) - L(f, P) = \sum_{i=1}^{n} \omega(f, [x_{i-1}, x_i])\Delta x_i$$

Proof. □

Theorem 3.2.3: Linearity

Let $f, g : [a, b] \to \mathbb{R}$ be integrable functions, then for any $\alpha \in \mathbb{R}$, the function $\alpha f + g : [a, b] \to \mathbb{R}$ is integrable and moreover

$$\int_a^b (\alpha f + g) = \alpha \int_a^b f + \int_a^b g$$

Proof. This is one of the situations were it is more convenient to use the Riemman integral rather than the Darboux integral. As they are equivalent, we are free to use whichever we like. In this case, the result follows almost identically to the proofs of the linearity limit laws when we use the Riemann integral. □

Theorem 3.2.4: Additivity of Domain

If $f : [a, b] \to \mathbb{R}$ is integrable on $[a, c]$ and on $[c, b]$ for some $c \in (a, b)$, then f is integrable and

$$\int_a^b f = \int_a^c f + \int_c^b f$$

Proof. We first show that f is integrable. Let $\varepsilon > 0$, as $f_1 := f|_{[a,c]}$ and $f_2 := f_{[c,b]}$ are integrable, there are partitions P_1 of $[a, c]$ and P_2 of $[c, b]$ such that

$$U(f_1, P_1) - L(f_1, P_1) < \frac{\varepsilon}{2} \quad \text{and} \quad U(f_2, P_2) - L(f_2, P_2) < \frac{\varepsilon}{2}$$

Define $P = P_1 \cup P_2$ which is a partition of $[a, b]$ by construction. It then follows that

$$\begin{aligned} U(f, P) - L(f, P) &= (U(f_1, P_1) + U(f_2, P_2)) - (L(f_1, P_1) + L(f_2, P_2)) \\ &= (U(f_1, P_1) - L(f_1, P_1)) + (U(f_2, P_2) - L(f_2, P_2)) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon \end{aligned}$$

Thus, f is integrable. Now let's prove the integral equality. We aim to use uniqueness as stated in Lemma 3.1.6. Let P be a partition of $[a, b]$, and set $\widetilde{P} = P \cup \{c\}$ so that we may write $\widetilde{P} = P_1 \cup P_2$ where P_1 and P_2 are partitions of $[a, c]$ and $[c, b]$ respectively. By Lemma 3.1.6 we know that

$$L(f_1, P_1) \leq \int_a^c f \leq U(f_1, P_1) \quad \text{and} \quad L(f_2, P_2) \leq \int_c^b f \leq U(f_2, P_2)$$

By adding these two inequalities together, and applying Lemma 3.1.3, we have

$$\begin{gathered} L(f_1, P_1) + L(f_2, P_2) \leq \int_a^c f + \int_c^b f \leq U(f_1, P_1) + U(f_2, P_2) \\ L(f, \widetilde{P}) \leq \int_a^c f + \int_c^b f \leq U(f, \widetilde{P}) \end{gathered}$$

Finally, as $L(f, P) \leq L(f, \widetilde{P}) \leq U(f, \widetilde{P}) \leq U(f, P)$, the result follows by Lemma 3.1.6. □

Proposition 3.2.5: Inheritance

If $f : [a, b] \to \mathbb{R}$ is an integrable function, then the restriction of f to any subinterval $[c, d] \subseteq [a, b]$ is integrable.

Proof. Let $[c, d] \subseteq [a, b]$ and $\varepsilon > 0$. As f is integrable, there is a partition $P = \{x_0, \ldots, x_n\}$ of $[a, b]$ such that $U(f, P) - L(f, P) < \varepsilon$. Let $Q = (P \cap [c, d]) \cup \{c, d\}$, namely the points in P contained in $[c, d]$ together with the endpoints of the subinterval. By construction, Q is a partition of $[c, d]$ and

$$\begin{aligned} U(f_{[c,d]}, Q) - L(f_{[c,d]}, Q) &\leq U(f, P \cup \{c, d\}) - L(f, P \cup \{c, d\}) \\ &\leq U(f, P) - L(f, P) \\ &< \varepsilon \end{aligned}$$

□

Theorem 3.2.6: Monotonicity

If $f, g : [a, b] \to \mathbb{R}$ are integrable functions such that $f(x) \leq g(x)$ for all $x \in [a, b]$, then

$$\int_a^b f \leq \int_a^b g$$

Proof. Define $h := g - f$, by construction, $h(x) \geq 0$ for all $x \in [a, b]$ and by linearity h is integrable. Let $P = \{x_0, \ldots, x_n\}$ be a partition of $[a, b]$. On each subinterval $[x_{i-1}, x_i] \subseteq [a, b]$, we have

$$\inf_{x \in [x_{i-1}, x_i]} f(x) \geq \inf_{x \in [a,b]} f(x) \geq 0 \quad \Longrightarrow \quad 0 \leq \sum_{i=1}^{n} \inf_{x \in [x_{i-1}, x_i]} f(x) = L(f, P)$$

Thus, as $L(f, P) \geq 0$ for all partitions P, it follows by using linearity once again that

$$0 \leq \sup_P L(f, P) = \int_a^b g = \int_a^b g - \int_a^b f \quad \Longrightarrow \quad \int_a^b f \leq \int_a^b g$$

□

Theorem 3.2.7: Subnormality

If $f : [a, b] \to \mathbb{R}$ is integrable, then $|f| : [a, b] \to \mathbb{R}$ is integrable and

$$\left| \int_a^b f \right| \leq \int_a^b |f|$$

Proof. To prove that $|f|$ is integrable, we can make use of Lemma 3.2.2. Let $\varepsilon > 0$, by the integrability of f there is a partition P of $[a, b]$ for which

$$U(f, P) - L(f, P) < \varepsilon$$

We claim that the same partition works for $|P$. Indeed, write $P = \{x_0, \ldots, x_n\}$, by Lemma 3.2.2

$$\begin{aligned} U(|f|, P) - L(|f|, P) = \sum_{i=1}^{n} \omega(|f|, [x_{i-1}, x_i])\Delta x_i &\leq \sum_{i=1}^{n} \omega(f, [x_{i-1}, x_i])\Delta x_i \\ &= U(f, P) - L(f, P) \\ &< \varepsilon \end{aligned}$$

Thus, $|f|$. To prove the integral inequality, note that $-|f(x)| \leq f(x) \leq |f(x)|$ for all $x \in [a, b]$. By using monotonicity and linearity, it follows that

$$-\int_a^b |f| \leq \int_a^b f \leq \int_a^b |f| \quad \implies \quad \left| \int_a^b f \right| \leq \int_a^b |f|$$ □

3.3 Sufficient Conditions for Integrability

In this section we will present several sufficient conditions for the Darboux integrability of a bounded function. It's worth noting that there does exist a necessary and sufficient condition for integrability, though we do not have the tools to state or prove it.

Proposition 3.3.1: Monotonicity

If $f : [a, b] \to \mathbb{R}$ is monotone, then f is integrable.

Proof. We can start by first assuming f is monotonically increasing and making a useful observation and . Given a partition $P = \{x_0, \ldots, x_n\}$ of $[a, b]$, on each subinterval $[x_{i-1}, x_i]$, as f is increasing, we have

$$\sup_{x \in [x_{i-1}, x_i]} f(x) = f(x_i) \quad \text{and} \quad \inf_{x \in [x_{i-1}, x_i]} f(x) = f(x_{i-1})$$

If we were to sum over the differnces between the supremum and infimum on each subinterval, all but $f(b)$ and $-f(a)$ would cancel out. Thet trick from here is to choose a partition whose length is about some quantity involving ε and $f(b) - f(a)$. Indeed, given $\varepsilon > 0$, pick a partition P of $[a, b]$ for which

$$\|P\| < \frac{\varepsilon}{f(b) - f(a)}$$

Note that if $f(b) = f(a)$, we have that f is constant and hence integrable, so we may assume that $f(b) > f(a)$. Write $P = \{x_0, \ldots, x_n\}$, then from our observation above, it follows that

$$\begin{aligned} U(f,P) - L(f,P) &= \sum_{i=1}^{n} \left(\sup_{x \in [x_{i-1}, x_i]} f(x) - \inf_{x \in [x_{i-1}, x_i]} f(x) \right) \Delta x_i \\ &\leq \|P\| \sum_{i=1}^{n} (f(x_i) - f(x_{i-1})) \\ &= \|P\|(f(b) - f(a)) \\ &< (f(b) - f(a)) \frac{\varepsilon}{f(b) - f(a)} \\ &= \varepsilon \end{aligned}$$

Finally, if f is monotonically decreasing, we can apply the above to $g := -f$ and use linearity. □

Theorem 3.3.2: Continuity

If $f : [a,b] \to \mathbb{R}$ is continuous, then f is integrable.

Proof. Let $\varepsilon > 0$, recall that a continuous function on a compact set is necessarily uniformly continuous, and so there is $\delta > 0$ such that for all $x, y \in [a,b]$

$$|x - y| < \delta \implies |f(x) - f(y)| < \frac{\varepsilon}{b - a}$$

Let P be a partition of $[a,b]$ such that $\|P\| < \delta$ and write $P = \{x_0, \ldots, x_n\}$. As f is continuous and each $[x_{i-1}, x_i]$ is compact, by the Extreme Value Theorem, there is $t_i, s_i \in [x_{i-1}, x_i]$ such that

$$\sup_{x \in [x_{i-1}, x_i]} f(x) = f(t_i) \quad \text{and} \quad \inf_{x \in [x_{i-1}, x_i]} f(x) = f(s_i)$$

Moreover, as $s_i, t_i \in [x_{i-1}, x_i]$, we have $|s_i - t_i| \leq \Delta x_i < \delta$ and thus

$$\begin{aligned} U(f,P) - L(f,P) = \sum_{i=1}^{n} (f(t_i) - f(s_i)) \Delta x_i &< \frac{\varepsilon}{b-a} \sum_{i=1}^{n} \Delta x_i \\ &= \frac{\varepsilon}{b-a}(b-a) \\ &= \varepsilon \end{aligned}$$

□

Proposition 3.3.3: Extension

If $f, g : [a,b] \to \mathbb{R}$ are bounded functions such that f is integrable on $[t,b]$ for all $t \in (a,b)$ and g is integrable on $[a,s]$ for all $s \in (a,b)$, then f and g are integrable and moreover

$$\int_a^b f = \lim_{t \to a^+} \int_t^b f \quad \text{and} \quad \int_a^b g = \lim_{s \to b^-} \int_a^s g$$

Proof. Let $\varepsilon > 0$, if $\omega := \omega(f, [a,b]) = 0$, then f is a constant function and the result follows immediately. Assuming $\omega \neq 0$, define

$$\delta = \min\left\{ \frac{\varepsilon}{2\omega}, \frac{b-a}{2} \right\} > 0$$

Set $x_1 = a + \delta$. Note that $x_1 \in (a, b)$ by construction, and so by assumption, f is integrable on $[x_1, b]$. It follows that there is a partition Q of $[x_1, b]$ such that

$$U(f, Q) - L(f, Q) < \frac{\varepsilon}{2}$$

Refine Q by $\{a\}$ so that $P := Q \cup \{a\}$ is a partition of $[a, b]$. By construction, it follows that

$$\begin{aligned} U(f, P) - L(f, P) &= \omega(f, [a, x_1])(x_1 - a) + U(f, Q) - L(f, Q) \\ &< \omega\delta + \frac{\varepsilon}{2} \\ &\leq \omega\left(\frac{\varepsilon}{2\omega}\right) + \frac{\varepsilon}{2} \\ &= \varepsilon \end{aligned}$$

We now show that the integral of f can be obtained by taking limits. Let $\varepsilon > 0$, as f is bounded, there is $M > 0$ such that $|f(x)| \leq M$ for all $x \in [a, b]$. Define $\delta = \frac{\varepsilon}{M} > 0$, given $t \in (a, a + \delta)$, by additivity of domain, subnormality and monotonicity, it follows that

$$\left|\int_a^b f - \int_t^b f\right| = \left|\int_a^t f\right| \leq \int_a^t |f| \leq M(t - a) < M\left(\frac{\varepsilon}{M}\right) = \varepsilon$$

Here we have taken for granted the straight forward result that $\int_a^b 1 = b - a$ for any $a < b \in \mathbb{R}$. An almost identical proof will establish the result for g. □

Corollary 3.3.4: Discontinuous Functions

If $f : [a, b] \to \mathbb{R}$ has finitely many points of discontinuity, then f is integrable.

Proof. We can combine the results of the previous two theorems. Enumerate the discontinuities of f as $\{d_1, \dots, d_n\}$ and write $a = d_0, b = d_{n+1}$. Then we can partition $[a, b]$ into subintervals of the form $[d_{i-1}, d_i]$. For such a subinterval, let $m_i = \frac{1}{2}(d_i + d_{i-1})$ denote the midpoint. For any $t \in (d_{i-1}, m_i)$, we have that f is continuous on $[t, m_i]$ and hence integrable, thus buy Proposition 3.3.3 it follows that f is integrable on $[d_{i-1}, m_i]$. Similarly, for any $s \in (m_i, d_i)$, we have that f is continuous on $[m_i, s]$ and hence integrable, and thus f is integrable on $[m_i, d_i]$ again by Proposition 3.3.3. Finally, as f is integrable on $[d_{i-1}, m_i]$ and $[m_i, d_i]$, it follows by additivity of domain that f is integrable on $[d_{i-1}, d_i]$ for each i, and thus by additivity of domain once again it follows that f is integrable on $[a, b]$. □

Proposition 3.3.5: Composition I

If $g : [a, b] \to \mathbb{R}$ is an integrable function and $f : [c, d] \to \mathbb{R}$ is a continuous function such that $g([a, b]) \subseteq [c, d]$, then $f \circ g : [a, b] \to \mathbb{R}$ is integrable.

Proof. Let $\varepsilon > 0$, if $\omega := \omega(f \circ g, [a, b]) = 0$, then $f \circ g$ is necessarily constant and hence integrable. If not, as $f : [c, d] \to \mathbb{R}$ is continuous and hence uniformly continuous, there is $\delta > 0$ such that

$$|x - y| < \delta \implies |f(x) - f(y)| < \frac{\varepsilon}{2(b - a)}$$

moreover as g is integrable there is a partition P of $[c,d]$ for which $U(g,P)-L(g,P) < \frac{\varepsilon\delta}{2\omega}$. Enumerate $P=\{x_0,\dots,x_n\}$ and set $I_k=[x_{k-1},x_k]$ for $k=1,\dots,n$. We can then write

$$U(f\circ g,P)-L(f\circ g)=\sum_{k=1}^{n}\omega(f\circ g)|I_k| = \sum_{\omega(g,I_k)<\delta}\omega(f\circ g)|I_k| + \sum_{\omega(g,I_k)\geq\delta}\omega(f\circ g)|I_k|$$

Suppose $k\in\{1,\dots,n\}$ such that $\omega(g,I_k)<\delta$, then by the uniform continuity of f we have

$$|g(x)-g(y)|<\delta \implies |(f\circ g)(x)-(f\circ g)(y)| < \frac{\varepsilon}{2(b-a)} =: \varepsilon^*$$

for all $x,y\in I_k$, and so $\omega(f\circ g,I_k)\leq\varepsilon^*$. Now we have to bound the contributions of the "bad" intervals, namely the ones where the oscillation may exceed δ. Note that

$$\delta(b-a)\geq\sum_{\omega(g,I_k)\geq\delta}|I_k|$$

□

Proposition 3.3.6: Composition II

Let $f:[a,b]\to\mathbb{R}$ be an integrable function and $g:[c,d]\to[a,b]$ a differentiable function such that $g'(x)\geq\xi>0$ for all $x\in[c,d]$. Then $f\circ g:[c,d]\to\mathbb{R}$ is integrable.

Proof. Let $\varepsilon>0$, as f is integrable on $[g(c),g(d)]=g([c,d])\subseteq[a,b]$ by Proposition 3.2.5, there is a partition $Q=\{y_0,\dots,y_n\}$ of $[g(c),g(d)]$ such that $U(f,Q)-L(f,Q)<\xi\varepsilon$. As g is monotone increasing and a surjection on its image $[g(c),g(d)]$, for each $y_i\in Q$, there is $x_i\in[c,d]$ such that $y_i=g(x_i)$, where $x_0=c$ and $x_n=d$. Thus, $P=\{x_0,\dots,x_n\}$ is a partition of $[c,d]$ and moreover as g is strictly increasing, for each subinterval $[x_{i-1},x_i]$, we can write

$$\sup_{x\in[x_{i-1},x_i]}(f\circ g)(x)=\sup_{x\in[g(x_{i-1},g(x_i)]}f(x) \quad\text{and}\quad \inf_{x\in[x_{i-1},x_i]}(f\circ g)(x)=\inf_{x\in[g(x_{i-1},g(x_i)]}f(x)$$

Using this, we can write the difference in the upper and lower sums as

$$\begin{aligned}U(f\circ g,P)-L(f\circ g,P)=\sum_{i=1}^{n}V_{f\circ g}([x_{i-1},x_i])\Delta x_i &= \sum_{i=1}^{n}V_f([g(x_{i-1}),g(x_i)])\Delta x_i\\ &=\sum_{i=1}^{n}V_f([y_{i-1},y_i)])\Delta y_i\cdot\frac{\Delta x_i}{\Delta y_i}\end{aligned}$$

Where $\Delta y_i\neq0$ as g is strictly increasing. By applying the mean value theorem to g on a subinterval $[x_{i-1},x_i]$, there is $\theta_i\in(x_{i-1},x_i)$ such that

$$\frac{g(x_i)-g(x_{i-1})}{x_i-x_{i-1}}=g'(\theta_i) \quad\implies\quad \frac{\Delta x_i}{\Delta y_i}=\frac{1}{g'(\theta_i)}\leq\frac{1}{\xi}$$

Finally, putting everything together, we can conclude

$$\begin{aligned}U(f\circ g,P)-L(f\circ g,P)=\sum_{i=1}^{n}V_f([y_{i-1},y_i)])\Delta y_i\cdot\frac{\Delta x_i}{\Delta y_i} &\leq\frac{1}{\xi}\sum_{i=1}^{n}V_f([y_{i-1},y_i)])\Delta y_i\\ &=\frac{1}{\xi}(U(f,Q)-L(f,Q))\\ &<\varepsilon\end{aligned}$$

□

3.4 The Fundamental Theorem of Calculus

Theorem 3.4.1: The First Fundamental Form

If $f : [a, b] \to \mathbb{R}$ is integrable, then the function $F : [a, b] \to \mathbb{R}$ defined by

$$F(x) = \int_a^x f(t)\,\mathrm{d}t$$

is uniformly continuous, and moreover if f is continuous at a point $c \in (a, b)$, then F is differentiable at c and $F'(c) = f(c)$.

Proof. □

Theorem 3.4.2: The Second Fundamental Form

If $f[a, b] \to \mathbb{R}$ is integrable and $F : [a, b] \to \mathbb{R}$ is a continuous antiderivative of f which is differentiable at all but finitely many points in $[a, b]$, then

$$\int_a^b f = F(b) - F(a)$$

Proof. Enumerate the discontinuities of F as $D = \{d_0, \dots, d_n\}$, and fix a partition $P = \{y_0, \dots, y_m\}$ of $[a, b]$. Refine P by D, namely set $\widetilde{P} = P \cup D$ and write $\widetilde{P} = \{x_0, \dots, x_N\}$. By construction, on each subinterval $[x_{i-1}, x_i]$, F is continuous and differentiable on (x_{i-1}, x_i). By the Mean Value Theorem, it follows that there is $\theta_i \in (x_{i-1}, x_i)$ such that

$$F(x_{i-1} - F(x_i) = F'(\theta_i)(x_i - x_{i-1}) = f(\theta_i)(x_i - x_{i-1})$$

As $\theta_i \in [x_{i-1}, x_i]$, we have $\inf_{x\in[x_{i-1},x_i]} f(x) \le f(\theta_i) \le \sup_{x\in[x_{i-1},x_i]} f(x)$, and so

$$L(f, \widetilde{P}) \le \sum_{i=1}^{N} f(\theta_i)(x_i - x_{i-1}) \le U(f, \widetilde{P})$$

$$L(f, P) \le L(f, \widetilde{P}) \le \underbrace{\sum_{i=1}^{N}(F(x_i) - F(x_{i-1}))}_{F(b)-F(a)} \le U(f, \widetilde{P}) \le U(f, P)$$

Thus, by uniqueness, the result follows. □

Example 3.4.3

Examples

4 Metric Space Topology

4.1 Open and Closed Sets

4.2 Sequences and Continuity

4.3 The Subspace Topology

4.4 Compactness

4.5 Connectedness

Definition 4.5.1: Disconnections and Connected Sets

We say that a set $S \subseteq \mathbb{R}$ is *disconnected* if there exists disjoint open sets $U, V \subseteq \mathbb{R}$ such that $S \subseteq U \cup V$ and $U \cap S, V \cap S \neq \varnothing$, such a pair (U, V) is called a *disconnection*. We say that a set is *connected* if it is not disconnected

Though this will be our standard definition (those who are more familiar with topology already may have noticed that we explicitly avoided mentioning a notion of the "subspace topology" in our definition of a disconnection), we can rephrase it using a simple property of sets. Recall that

$$(S \cap U) \cup (S \cap V) = S \cap (U \cup V)$$

and so $S \subseteq U \cup V$ if and only if $S = S \cap (U \cup V)$, namely $S \subseteq \mathbb{R}$ is disconnected if and only if there exists disjoint open sets $U, V \subseteq \mathbb{R}$ such that $S \cap U, S \cap V \neq \varnothing$ and $(S \cap U) \cup (S \cap V) = S$.

Definition 4.5.2: Formal Intervals

A set $I \subseteq \mathbb{R}$ is called an *interval* if for every $a, b \in I$ and every $x \in \mathbb{R}$, if $a < x < b$ then $x \in I$.

Though it may seem a bit strange and unnecessary at first, with a bit of thought hopefully this definition should make sense. We say that a set I an interval if for any two distinct points $x, y \in I$, all real numbers between x and y are also in I. Let's see a simple example that will lay some groundwork for the next result. Consider the set $A = [1, 2] \cup [4, 5]$. This is a set that we know is not an interval, and if we were to draw it on a number line, it would make sense that it shouldn't be considered a connected set (it's the union of two separated pieces). Let's verify that A is not connected by explicitly constructing a disconnection. Note that $2, 4 \in A$, yet $3 \notin A$, showing that A is in fact not an interval. Additionally, by considering the open sets $U = (-\infty, 3)$ and $V = (3, \infty)$, it follows that (U, V) forms a disconnection of A. The exact details are generalized to form the following Proposition.

Proposition 4.5.3

If $I \subseteq \mathbb{R}$ is not an interval, then I is disconnected.

Proof. As I is not an interval, negating Definition 4.5.2 says there is $a < b \in I$ and $x \in \mathbb{R}$ such that $a < x < b$ and $x \notin I$. Define $U = (-\infty, x), V = (x, \infty)$. Then clearly U and V are open an disjoint, moreover $a \in U \cap I, b \in V \cap I$ and as $x \notin I$ we have $I \subseteq \mathbb{R} \setminus \{x\} = U \cup V$. Thus (U, V) form a disconnection and hence I is disconnected. $\square$

Proposition 4.5.4

If $I \subseteq \mathbb{R}$ is disconnected, then I is not an interval.

Proof. Suppose $I \subseteq \mathbb{R}$ admits a disconnection (U, V). Pick $x \in U \cap I, y \in V \cap I$, as $U \cap V = \varnothing$ we have $x \neq y$ and so assume without loss of generality $x < y$. Define

$$A = \{a \geq x : [x, a] \cap I \subseteq U \cap I\}$$

Note that $x \in A$ and as $y \notin I \cap U$ we have y is an upper bound for A, thus by completeness $c = \sup(A)$ exists. We show $c \notin I$. Suppose $c \in U \cap I$, then as $c \in U$ and U is open, there is $r > 0$ such that $(c - r, c + r) \subseteq U$ and hence $(c - r, c + r) \cap I \subseteq U \cap I$. As $c - r$ is not an upper bound, there is $a_0 \in A$ such that $c - \frac{r}{2} < a_0$, and hence as $[x, a_0] \cap I \subseteq U \cap I$ we have

$$\left[x, c + \frac{r}{2}\right] \cap I = \left(\left[x, c - \frac{r}{2}\right] \cap I\right) \cup \left(\left[c - \frac{r}{2}, c + \frac{r}{2}\right] \cap I\right) \subseteq U \cap I$$

Namely, $c + \frac{r}{2} \in A$, contradicting that $c = \sup(A)$ is an upper bound, and so $c \notin U \cap I$. Conversely, suppose $c \in V \cap I$. Again, as $c \in V$ and V is open, there is $r > 0$ such that $(c - r, c + r) \cap I \subseteq V \cap I$. Finally, as $(c - r, c - \frac{r}{2}] \cap I \subseteq (c - r, c + r) \cap I \subseteq V \cap I$, we have

$$\left[x, c - \frac{r}{2}\right] \cap I \cap (V \cap I) \neq \varnothing \quad \implies \quad \left[x, c - \frac{r}{2}\right] \cap I \not\subseteq U \cap I$$

Namely $c - \frac{r}{2} \notin A$ and it follows that $c - \frac{r}{2}$ is an upper bound for A, contradicting that $c = \sup(A)$ is the least upper bound. Thus, $c \notin V \cap I$ and hence $c \notin I$ as $I = (U \cap I) \cup (V \cap I)$. Thus, as $x \in A$ and y is an upper bound for A we have $x \leq c \leq y$ and moreover as $x \in U \cap I, y \in V \cap I$, we have $x < c < y$. Thus, I is not an interval. $\square$

By combining the previous two propositions, we have that $I \subseteq \mathbb{R}$ is disconnected if and only if I is not an interval, where the contrapositive says that a set is connected in $\mathbb{R}$ if and only if it is an interval.

Proposition 4.5.5

Let $S \subseteq \mathbb{R}$ and $f : S \to \mathbb{R}$ a continuous function. If $f(S)$ is disconnected then S is disconnected.

Proof. Let (U, V) be a disconnection of $f(S)$. By the continuity of f, as $U, V \subseteq \mathbb{R}$ are open we have that $f^{-1}(U), f^{-1}(V)$ are open. We claim $(f^{-1}(U), f^{-1}(V))$ forms a disconnection of S. Indeed, note that $f^{-1}(U) \cap f^{-1}(V) = f^{-1}(U \cap V) = \varnothing$ as $U \cap V = \varnothing$, moreover

$$f^{-1}(U) \cup f^{-1}(V) = f^{-1}(U \cup V) \supseteq f^{-1}(f(S)) \supseteq S$$

as $f(S) \subseteq U \cup V$. Finally, Let $y \in f(S) \cap U$, then writing $y = f(x)$ for $x \in S$ we have $x \in S \cap f^{-1}(U)$ and so $f^{-1}(U) \cap S \neq \varnothing$ and similarly $f^{-1}(V) \cap S \neq \varnothing$. Thus, $(f^{-1}(U), f^{-1}(V))$ forms a disconnection of S and hence S is disconnected. $\square$

Corollary 4.5.6: Intermediate Value Theorem

If $f : [a, b] \to \mathbb{R}$ is a continuous function such that $f(a)f(b) < 0$, then there is $c \in (a, b)$ such that $f(c) = 0$.

Proof. As $[a, b] \subseteq \mathbb{R}$ is connected and f is continuous, by Proposition 4.5.5 $f([a, b])$ is connected. Assume without loss of generality $f(a) < 0 < f(b)$. Then as $f([a, b])$ is an interval we have $0 \in f([a, b])$ by Definition 4.5.2 and the result follows. $\square$